JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **143**, 587–596 (1989).

Stability Properties of Positively Invariant Linear Discrete Time Systems

C. BURGAT

Laboratoire d'Automatique et d'Analyse des Systèmes du C.N.R.S., 7 Avenue du Colonel Roche, 31077 Toulouse Cédex, France

AND

A. BENZAOUIA

Laboratoire d'Automatique et d'Analyse des Systèmes du C.N.R.S., 7 Avenue du Colonel Roche, 31077 Toulouse Cédex, France, and Département de Physique, Faculté des Sciences, Université Cadi, Ayyad, Marrakesh, Morocco

Submitted by S. M. Meerkov

Received November 22, 1987

The purpose of this paper is to characterize the stability properties (i.e., asymptotic stability, critical stability, instability) of positively invariant discrete time linear systems described by $x_{k+1} = Ax_k$. Necessary and sufficient conditions are given in geometrical form using cones; such a characterization does not use the classical knowledge of the spectral radius of matrix A. In case of nonnegative matrices, the connections between the proposed results and the theory of M-matrices are pointed out. © 1989 Academic Press, Inc.

1. INTRODUCTION

This paper is devoted to the geometric characterization of stability properties for positively invariant discrete time systems described by

$$x_{k+1} = Ax_k, \qquad x_k \in R^n, \tag{1}$$

where A is a real $n \times n$ matrix, $k \in J_+$ the set of nonnegative integers. In (1) and throughout this paper it is assumed that matrix A leaves a proper cone $K_+ \subset R^n$ positively invariant, that is

$$AK_+ \subseteq K_+. \tag{2}$$

0022-247X/89 $3.00

The results presented in this paper generalize those given in [1]. In [1], matrix A of (1) is nonnegative and $K_+ = R^n_+$, i.e., K is a simplicial cone; further the asymptotic stability property of (1) is the only one to be characterized. We now consider the case where K_+ is a proper cone, i.e., not necessarily simplicial or polyhedral, and matrix A is not required to be nonnegative.

The aim of this paper is to characterize the stability properties of System (1), that is, its asymptotic stability, critical stability, and instability, without the classical use of the spectral radius $\rho(A)$ of matrix A. This can now be accomplished because the system described in (1) possesses Property (2), which is an intrinsic property of motions of (1). Necessary and sufficient conditions expressed by means of cones, directly related to the positively invariant cone K_+, can be obtained.

Over the last twenty years, much emphasis has been placed on matrices leaving a cone positively invariant, e.g., Birkoff [2], Krein and Rutman [3], Vandergraft [4]; see also Berman and Plemmons [5] for a bibliographic background. In the field of positively invariant dynamical systems, some papers investigate the specific properties of such a class of systems, e.g., Yorke [6], Stern [7]. They deal only with the case of continuous time systems. However, the results by these authors are not analogous to those presented in this paper.

The proposed results are of interest in the field of control theory and especially for the study of the stability properties of linear discrete time systems with constrained controls. In this case, these results allow us to construct some dissymmetrical sets which possess both positive invariance and asymptotic stability properties, without the use of Lyapunov functions [11].

This paper is organized as follows. In Section 2 we recall some useful definitions concerning stability and cones. Section 3 deals with both translated cones relatively to their invariance property and the basic lemmas that are required to demonstrate the main results of Section 4. In Section 5, we point out the connections of the previous results with both the polyhedral cones associated to M-matrices and the stability properties of dynamical systems for which the matrix A is nonnegative.

2. Definitions

In this part we recall some definitions required in the sequel.

Definition 2.1. The equilibrium $x = 0$ of the dynamical system (1) is said to be

— stable, if for each $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that if $\|x_0\| < \delta$, then $\|x(k; x_0)\| < \varepsilon$, $\forall k \in J_+$;

— asymptotically stable if it is stable and further if $\|x(k; x_0)\| \to 0$ when $k \to \infty$;

— unstable, if it is not stable;

— critically stable, if the spectral radius $\rho(A) = 1$. In this case the equilibrium $x = 0$ can be stable or unstable, as specified in Remark 2.2.

Remark 2.2. For linear discrete time systems (1) stability properties are well characterized by means of $\rho(A)$. It is well known that $\rho(A) < 1$ implies the asymptotic stability property, whereas $\rho(A) > 1$ induces instability. For a critically stable system (1), $\rho(A) = 1$, we obtain the stability property, in the sense of Definition 2.1, if and only if eigenvalues of matrix A with modulus 1 are simple; then any other spectral case with $\rho(A) = 1$ corresponds to the instability of the equilibrium [8].

DEFINITION 2.3. A set K in real Euclidean n-space R^n is said to be a cone if [10]

(i) K is nonempty,

(ii) K is a closed subset of R^n,

(iii) $K + K \subseteq K$,

(iv) $\alpha K \subseteq K$ for all $\alpha > 0$,

(v) $\operatorname{Int} K \neq \emptyset$,

(vi) $K \cap (-K) = \{0\}$.

From its definition a solid cone has a topological interior point (property (v)). From (vi), K is a pointed cone. A pointed and solid cone is termed proper cone. Moreover Property (iii) implies the convexity of K. In the sequel we also denote ∂K the boundary of K.

DEFINITION 2.4. A nonempty set $K \subset R^n$ is said to be positively invariant by a matrix A if $\forall x \in K$, $Ax \in K$; that is, $AK \subseteq K$.

Such a definition may be applied directly to the case of dynamical system (1) in relation to its motions $x(k; x_0) = A^k x_0$, where $x_0 \in R^n$ is an initial state.

DEFINITION 2.5. (i) A nonempty subset $K \subset R^n$ is said to be positively invariant (with respect to motions of (1)) if $\forall x_0 \in K$, $x(k; x_0) \in K$, $\forall k \in J_+$;

(ii) A nonempty subset $K \subset R^n$ is said to be positively invariant and asymptotically stable if $\forall x_0 \in K$, $x(k; x_0) \in K$, $\forall k \in J_+$, and further, if $x(k; x_0) \to 0$ as $k \to \infty$.

3. Preliminary Results

The purpose of this present section is, first, to obtain a characterization of the positive invariance property for a class of sets generated from the translation of a proper cone K and defined as

$$K+x=\{z\in R^n \mid z=y+x,\ y\in K,\ x\in R^n\}.$$

In the following we denote $K_+=K$ and $K_-=-K_+$. From the matrix Π given by $\Pi=A-\mathbb{1}$, where $\mathbb{1}$ is the identity matrix in R^n, we define the set C_+ as

$$C_+=\{x\in R^n \mid -\Pi x=K_+\},\qquad C_-=-C_+.$$

LEMMA 3.1. *Let the system* (1) *have Property* (2); *then the set* K_-+x (*resp.* K_++x) *has the properties*

(i) $A(K_-+x)\subset K_-+x$ (*resp.* $A(K_++x)\subset K_++x$) *if and only if* $-\Pi x\in K_+$ (*resp.* $+\Pi x\in K_+$);

(ii) $A(K_-+x)\subset \operatorname{Int}(K_-+x)$ (*resp.* $A(K_++x)\subset \operatorname{Int}(K_++x)$) *if and only if* $-\Pi x\in \operatorname{Int} K_+$ (*resp.* $+\Pi x\in \operatorname{Int} K_+$).

Proof. (i) (If) Let $y\in K_-+x$, then $y=z+x$ with $z\in K_-$; we get $Ay=Az+\Pi x+x$. From Condition (i) of Lemma 3.1, $-\Pi x\in K_+$, and with Property (2), $Az\in K_+$; it follows that $Ay\in K_-+x$ $\forall y\in K_-+x$.

(Only if) Let us assume that $-\Pi x\in K_+$ and $Ay\notin K_-+x$, whereas $y\in K_-+x$. Let $y=x$, $y\in K_-+x$. We get $Ay=\Pi x+x\in K_-+x$, contradicting the assumption $Ay\notin K_-+x$.

The proof of Part (ii) readily follows from the latter.

Remark 3.2. It is interesting to remark that Condition (ii) of Lemma 3.1 does not necessitate the K-irreducibility property in Vandergraft's sense [4]; that is, $A\ (K_+\backslash\{0\})\subset \operatorname{Int} K_+$.

Let us now recall a useful result concerning the spectral properties of matrices which have invariant proper cones.

LEMMA 3.3 [3]. *If K is a proper cone such that $AK\subseteq K$, then*

(i) *$\rho(A)$ is an eigenvalue,*

(ii) *the degree of $\rho(A)$ is not smaller than the degree of any other eigenvalue having the same modulus,*

(iii) *K contains an eigenvector corresponding to $\rho(A)$.*

In the sequel of this section, Lemmas 3.4 and 3.5 treat the specific properties of System (1) with Property (2), without making any other assumption on stability properties of (1). At the opposite, Lemmas 3.6 to 3.9 are directly connected with the spectral radius $\rho(A)$, therefore with the stability properties of system (1).

LEMMA 3.4. *Let System* (1) *have property* (2); *then the following properties hold*:

(i) $\forall x \in K_+$, $-\Pi x \in K_- + x$,

(ii) $\forall x \in \partial K_+$, $-\Pi x \notin \operatorname{Int} K_+$ (*resp.* $\forall x \in \partial K_-$, $+\Pi x \notin \operatorname{Int} K_+$).

Proof. Part (i) is obvious. Part (ii): Let $x \in \partial K_+$ and the translated cone $K_- + x$. Let us assume that $-\Pi x \in \operatorname{Int} K_+$; thus for each $y \in K_- + x$ and from Lemma 3.1(ii) follows $Ay \in \operatorname{Int}(K_- + x)$. Then, for $x \in \partial K_+$, $x \in K_- + x$, we get $Ax \in \operatorname{Int}(K_- + x)$. But this is impossible, since using Property (2) $Ax \in K_+$ and $\operatorname{Int}(K_- + x) \cap K_+ \neq \varnothing$. Hence Property (ii) of the lemma follows.

LEMMA 3.5. *If* C_+ (*resp.* C_-) *is a nonempty cone, then* $AC_+ \subset C_+$ (*resp.* $AC_- \subset C_-$).

Proof. Let C_+ be defined by (4) and $x \in C_+$; thus $-\Pi x \in K_+$. Then, with Property (2), $-A\Pi x \in K_+$ or $-\Pi A x \in K_+$. Thus, $\forall x \in C_+$ we have $Ax \in C_+$.

LEMMA 3.6. *If* $\rho(A) = 1$ *then*

$$\nexists y \in K_+ \qquad \textit{such that} \quad -\Pi y \in \operatorname{Int} K_+ \ (\textit{resp.}\ +\Pi y \in \operatorname{Int} K_+)$$

or equivalently,

$$\forall y \in K_+, \ -\Pi y \notin \operatorname{Int} K_+ \ (\textit{resp.}\ +\Pi y \notin \operatorname{Int} K_+).$$

Proof. Since K_+ is a solid cone, and from Lemma 3.3, there exists a vector $y_0 \in K_+$ such that $Ay_0 = y_0$ $(\rho(A) = 1)$. Let H be a hyperplane such that 0, $y_0 \in H$. Without loss of generality, assume that $y_0 \in \operatorname{Int} K_+$. Let $z, t \in K_+ \cap H$ be chosen so that $y_0 \in \partial(K_+ + z)$, $y_0 \in \partial(K_- + t)$. Assume now that $-\Pi t \in \operatorname{Int} K_+$ and $+\Pi z \in \operatorname{Int} K_+$; thus from Lemma 3.1(ii) we should have $\forall y \in \partial(K_- + t)$, $Ay \in \operatorname{Int}(K_- + t)$, and $\forall y \in \partial(K_+ + z)$, $Ay \in \operatorname{Int}(K_+ + z)$. But this is impossible since $y_0 \in \partial(K_- + t)$, $y_0 \in \partial(K_+ + z)$, and $Ay_0 = y_0$; this contradicts the assumption.

We repeat this process for all $z, t \in K_+ \cap H$ and for all hyperplanes H such that 0, $y_0 \in H$, Lemma 3.6 follows.

LEMMA 3.7. *If* $\rho(A) < 1$ *then*

(i) C_+ *is a solid cone*,

(ii) $(C_+ \cap K_+)\backslash\{0\} \neq \varnothing$.

Proof. (i) If $\rho(A) < 1$ then $\lambda_i(A) \in \mathscr{C}(0, \rho)$, $\mathscr{C}(0, \rho)$ being the closed circle of radius ρ centred at the origin. Therefore $\lambda_1(\Pi) \in \mathscr{C}(-1, \rho)$ and $0 \notin \mathscr{C}(-1, \rho)$. Hence, $\lambda_i(\Pi) \neq 0$ $\forall i$ and Π is a nonsingular matrix. From [9], C_+ is a cone and, further, C_+ is a solid cone since Π is nonsingular.

(ii) From Lemma 3.3, there exists $y_0 \in K_+$, $y_0 \neq 0$ such that $Ay_0 = \rho(A)\, y_0$; thus $-\Pi y_0 = (1 - \rho(A))\, y_0$. Since $\rho(A) < 1$, then $-\Pi y_0 = \mu y_0$, $\mu > 0$ and $y_0 \in C_+$; it follows that $(K_+ \cap C_+)\backslash\{0\} = \varnothing$.

LEMMA 3.8. *If* $\rho(A) < 1$, *then there exists* $z \in \operatorname{Int} K_+$ *such that* $-\Pi z \in \operatorname{Int} K_+$.

Proof. Let us assume that $\nexists z \in \operatorname{Int} K_+$ such that $-\Pi z \in \operatorname{Int} K_+$. Hence, $\forall z \in \operatorname{Int} K_+$ we must satisfy $-\Pi z \in R^n\backslash\operatorname{Int} K_+$. Let a vector y_0 such that $Ay_0 = \rho(A)\, y_0$ (Lemma 3.3) be chosen so that $y_0 \in \partial(K_- + z)$, $y_0 \neq z$, $z \in \operatorname{Int} K_+$. Since $\rho(A) < 1$, it follows that $Ay_0 \in (K_- + z)$. Lemma 3.1 suggests, then, at least that $-\Pi z \in K_+$, since the case $-\Pi z \in \operatorname{Int} K_+$ is excluded by assumption. Since $-\Pi z \in K_+ \cap (R^n\backslash\operatorname{Int} K_+)$, we derive $-\Pi z \in \partial K_+$. Since $y_0 \in \partial(K_- + z)$ we then can write $y_0 = z + u$, $u \in \partial K_-$, and then $Ay_0 = z + \Pi z + Au$. From Lemma 3.6, it follows that $Au \notin \operatorname{Int} K_-$, therefore $Au \in R^n\backslash\operatorname{Int} K_-$; but $\Pi z \in \partial K_-$ gives $\Pi z + Au \in R^n\backslash\operatorname{Int} K_-$. Consequently, $Ay_0 \in z + (R^n\backslash\operatorname{Int} K_-)$, which is impossible since $Ay_0 \in z + \operatorname{Int} K_+$. The assumption is contradicted, thus $\exists z \in \operatorname{Int} K_+$ such that $-\Pi z \in \operatorname{Int} K_+$.

LEMMA 3.9. *If* $\rho(A) < 1$ *then* $C_+ \subset K_+$.

Proof. From Lemma 3.8, $\exists z \in \operatorname{Int} K_+$ such that $-\Pi z \in \operatorname{Int} K_+$, that is, $z \in \operatorname{Int} C_+$. Using Lemma 3.7 we get $\operatorname{Int}(K_+ \cap C_+) \neq \varnothing$. Let us now assume that there exists $z_1 \in \operatorname{Int} C_+$ such that $z_1 \in R^n\backslash K_+$; in such a case $\exists z_2 \in \partial K_+$, $z_2 \in \operatorname{Int} C_+$. But Lemma 3.4(ii) implies $-\Pi z_2 \notin \operatorname{Int} K_+$; this contradicts the fact that $z_2 \in \operatorname{Int} C_+$. Indeed, in such a case we should have $-\Pi z_2 \in \operatorname{Int} K_+$; hence the contradiction. Therefore $\nexists z_1 \in R^n\backslash K_+$ such that $z_1 \in \operatorname{Int} C_+$. From Lemmas 3.7 and 3.8 we conclude that $\operatorname{Int} C_+ \subset \operatorname{Int} K_+$, $C_+ \subset K_+$.

4. MAIN RESULTS

THEOREM 4.1. *The equilibrium* $x = 0$ *of the dynamical system* (1), *with* (2), *is asymptotically stable if and only if*

(i) C_+ *is a solid cone,*

(ii) $C_+ \subset K_+$.

Proof. (If) If C_+ is a solid cone, then from Lemmas 3.3 and 3.4, $\exists y_0 \in C_+$ such that $Ay_0 = \rho(A)\, y_0$, or $-\Pi y_0 = (1-\rho(A))\, y_0$. Since $C_+ \subset K_+$, then $y_0 \in K_+$, $y_0 \in C_+$. Therefore, from $-\Pi y_0 = (1-\rho(A))\, y_0 \in K_+$, we have $\rho(A) < 1$; hence the asymptotic stability property of $x = 0$.

(Only if) Let us assume that $\rho(A) < 1$ and $C_+ \not\subset K_+$. This is in contradiction to Lemma 3.9. The same conclusion follows from the assumptions $\rho(A) < 1$ and C_+ nonsolid, from Lemma 3.7.

THEOREM 4.2. *The dynamical system* (1), *with* (2), *is critically stable* $(\rho(A) = 1)$ *if and only if*

$$((C_+ \cap C_-)\backslash\{0\}) \cap K_+ \neq \varnothing.$$

Proof. (If) Let $z \in C_+ \cap C_-$, $z \neq 0$; from Definition (4), we obtain $-\Pi z = y \in K_+$ and $+\Pi z = t \in K_+$. Thus $0 = y + t \in K_+$; it follows that $y = t = 0$, because of the pointedness property of K_+ due to the fact that K_+ is a proper cone. Hence, $-\Pi z$, $+\Pi z = 0$ which implies $Az = z$; then $\rho(A) = 1$. Indeed, $|\lambda_i(A)| \leqslant 1$ and the system is critically stable.

(Only if) Let us assume that $\rho(A) = 1$ and $(C_+ \cap C_-)\backslash\{0\} \notin K_+$. Since $\rho(A) = 1$ and from Lemma 3.3, $\exists y_0 \in K_+$, $y_0 \neq 0$, such that $Ay_0 = y_0$ or $\Pi y_0 = 0$. For such an eigenvector, $y_0 \in C_+$, $y_0 \in C_-$; one has $y_0 \in (C_+ \cap C_-)\backslash\{0\}$, which violates the previous assumption $(C_+ \cap C_-)\backslash \{0\} \notin K_+$.

Remark 4.3. From the preceding proof it follows that $C_+ \cap C_- = \operatorname{Ker} \Pi$.

THEOREM 4.4. *The equilibrium* $x = 0$ *of the dynamical system* (1), *with* (2), *is unstable* $(\rho(A) > 1)$ *if and only if*

$$(C_+ \cap K_+)\backslash\{0\} = \varnothing.$$

Proof. (If) Let us define a neighborhood of the origin by the set $\Lambda = K_- + \varepsilon \cap K_+ - \varepsilon$, where $\varepsilon \in K_+\backslash\{0\}$. In view of the condition of the theorem $(C_+ \cap K_+)\backslash\{0\} = \varnothing$; so there is no $\varepsilon \neq 0$, $\varepsilon \in K_+\backslash\{0\}$, such that $-\Pi\varepsilon \in K_+$. Thus, from Lemma 3.1, we conclude that Λ is not a positively invariant set. Then there always exists $\lambda > 1$ such that $Ay_0 \in \partial(K_+ - \lambda\varepsilon \cap K_- + \lambda\varepsilon)$. Iterating this process for $y_k = A^k y_0$ we can conclude to the instability.

(Only if) Let us assume that $\rho(A) > 1$ and $(C_+ \cap K_+)\backslash\{0\} \neq \varnothing$.

Hence $\exists \varepsilon \in C_+$, $\varepsilon \neq 0$, $\varepsilon \in K_+$ such that $-\Pi\varepsilon \in K_+$, which violates the assumption $\rho(A) > 1$ since from Theorems 4.2 and 4.3 we get $-\Pi\varepsilon \in K_+$, $\varepsilon \in K_+ \setminus \{0\}$, $\varepsilon \neq 0$ which implies $\rho(A) \leqslant 1$.

Theorems 4.1, 4.2, 4.4 give geometrical equivalence conditions corresponding to $\rho(A) < 1$, $\rho(A) = 1$, $\rho(A) > 1$, respectively. But for the critical case $\rho(A) = 1$ we must now geometrically characterize under what conditions the equilibrium is stable or unstable.

LEMMA 4.5. *If $\rho(A) = 1$ and System* (1) *is unstable then there does not exist $y \in \operatorname{Int} K_+$ such that $y \in C_+$.*

Proof. If the equilibrium is unstable there cannot exist a positively invariant set $\Lambda = (K_+ - y) \cap (K_- + y)$, $\{0\} \subset \operatorname{Int} \Lambda$, $y \in \operatorname{Int} K_+ \setminus \{0\}$.

If not, we obtain the stability property in the sense of Definition 2.1. From Lemma 3.1 this would be possible if and only if $-\Pi y \in K_+$, i.e., if $y \in C_+$. We conclude that if System (1) is unstable there cannot exist any $y \in \operatorname{Int} K_+ \setminus \{0\}$ such that $y \in C_+$.

THEOREM 4.6. *The equilibrium of a critically stable system* (1) *is*

(i) *stable if and only if $Z \cap \operatorname{Int} K_+ \neq \varnothing$,*

(ii) *unstable if and only if $Z \cap \operatorname{Int} K_+ = \varnothing$,*

where $Z = (C_+ \cap C_-) \setminus \{0\}$.

Proof. (i) (If) If the given condition is satisfied then, from Theorem 4.2, there exists $y \in Z$, $y \in \operatorname{Int} K_+ \setminus \{0\}$ such that $-\Pi y = 0 \in K_+$. It follows from Lemma 3.1 that the set Λ defined above is positively invariant and $\Lambda \supset \{0\}$; consequently, the equilibrium $x = 0$ is stable in the sense of Definition 2.1.

(Only if) Let us assume the condition of (i) is satisfied but the equilibrium $x = 0$ is unstable. With the condition of (i) and as in the (If) part, there exists $y \in \operatorname{Int} K_+ \setminus \{0\}$ such that $-\Pi y = 0 \in K_+$; that is $y \in C_+$. Lemma 4.5 leads to a contradiction.

(ii) In the critically stable case, the equilibrium is unstable if it is not stable; then the condition of (ii) readily follows.

5. APPLICATION

In this part we indicate how some classical results concerning the dynamical system (1) may be deduced from the previously established results, particularly when the cone K_+ is assumed to be simplicial.

First, let us write Theorem 4.1 in another form.

COROLLARY 5.1. *The equilibrium* $x=0$ *of System* (1) *with* (2) *is asymptotically stable if and only if*

$$\exists v \in \operatorname{Int} K_+ \text{ such that } -\Pi v \in \operatorname{Int} K_+.$$

Proof. It readily follows from Theorems 4.1, 4.2, and 4.3.

Let us now assume that K_+ is a simplical cone, i.e., defined as

$$K_+ = \{x \in \mathbb{R}^n_+ \mid Fx \geqslant 0\}, \qquad FK_+ = \mathbb{R}^n_+) \tag{3}$$

where F is a nonsingular matrix. This latter additional assumption leads to the following.

COROLLARY 5.2. *The equilibrium* $x=0$ *of System* (1) *with* (2) *and* (3) *is asymptotically stable if and only if*

(i) $\exists y > 0$ *such that* $-F\Pi F^{-1}y > 0$,

or equivalently

(i′) $-F\Pi F^{-1}$ *is an M-matrix.*

Proof. (i) From Corollary 5.1 and (3) we get $y = Fv > 0$, $v \in \operatorname{Int} K_+$, and $-F\Pi v > 0$ or $-F\Pi F^{-1}y > 0$.

(i′) From the mapping $y_k = Fx_k$ we obtain from (1) $y_{k+1} = FAF^{-1}y_k$, $y_{k+1} = \tilde{A}y_k$. But from (2) and (3) it follows that $\tilde{A}\mathbb{R}^n_+ \subset \mathbb{R}^n_+$, which implies that $\tilde{A}$ is a nonnegative matrix. Clearly since $-F\Pi F^{-1} = \tilde{A} - \mathbb{1}$, Condition (i) is equivalent to $-F\Pi F^{-1}$ is an M-matrix [5].

If we now assume that A is a nonnegative matrix, then $K_+ = \mathbb{R}^n_+$ is a positively invariant and simplicial cone for A. We finally get the following corollary.

COROLLARY 5.3. *If* A *is a nonnegative matrix, then System* (1) *is asymptotically stable if and only if one of the following equivalent forms holds*:

(i) $C_+(\Pi) \subset \mathbb{R}^n_+$;

(ii) $\exists v > 0$ *such that* $-\Pi v > 0$;

(iii) $-\Pi$ *is an M-matrix.*

Proof. Part (i) directly follows from Theorem 4.1 and parts (ii) and (iii) from Corollary 5.2.

It is of interest to note that the geometrical condition given by (i), as well as condition (iii), is given in [1, Theorem 2.3]. The condition (ii) is only a well known property characterizing an M-matrix [5].

CONCLUSION

Our results complete and unify numerous results present in the literature concerning the linear dynamical systems which possess a positively invariant cone. Furthermore all the stability properties are geometrically characterized by means of necessary and sufficient conditions; that is, not only asymptotic stability, but also critical stability and unstability. Additional assumptions on properties of cone K and matrix A, i.e., K_+ simplicial and A nonnegative, lead to classical results as particular cases.

REFERENCES

1. J. CHEGANCAS AND C. BURGAT, Polyhedral cone associated to M-matrices and stability of time varying discrete time systems, *J. Math. Anal. Appl.* **118** (1986), 88–96.
2. G. BIRKOFF, Linear transformations with invariant cones, *Amer. Math. Monthly* **72** (1967), 274–276.
3. M. G. KREIN, AND M. A. RUTMAN, Linear operators leaving invariant a cone in a Banach space, *Amer. Math. Soc. Transl. Ser. 1* **10** (1962), 199–325.
4. J. S. VANDERGRAFT, Spectral properties of matrices which have invariant a cone, *SIAM J. Appl.* **16** (1968), 1208–1222.
5. A. BERMAN AND R. J. PLEMMONS, "Nonnegative Matrices in the Mathematical Sciences," Academic Press, New York, 1979.
6. J. A. YORKE, Invariance for ordinary differential equations, *Math. Systems Theory* **3** (1967), 353–372.
7. R. J. STERN, A note on positively invariant cones, *Appl. Math. Systems Theory* **9** (1982), 67–72.
8. J. P. LASALLE, "The Stability of Dynamical Systems," SIAM Regional Conferences Series in Applied Mathematics, Soc. Industrial Appl. Math., Philadelphia, 1976.
9. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1972.
10. H. SCHNEIDER, AND M. VIDYASAGAR, Cross-positive matrices. *SIAM J. Numer. Anal.* **9** (1970), 508–519.
11. C. BURGAT AND A. BENZAOUIA, Positively invariants sets of discrete time systems with constrained controls, submitted for publication.